African Scientist Vol. 11, No. 1 March 31, 2010
Printed in Nigeria

1595-6881/2010 $12.00 + 0.00

http://www.klobex.org/afs

AFS 2009130/11106

Mathematical Modeling of Blood Flow Through A Constricted Vessel

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(Received November 9, 2009)

ABSTRACT: This paper designed a Mathematical Model which provides a good description of blood flow regimes via constricted and unconstricted vessels that induces high blood pressure. It also presents a solution which will predict the rates of blood flow and the pressure profiles across the domain.

Keywords: Blood Flow, Constricted vessel, Blood Pressure, Reynolds number

Introduction

Problems associated with blood flow in the system are numerous. Such problems can tend to make the heart malfunction which lead to a mild or severe disease condition. One of such problem is the high blood pressure (hypertension). Blood flow to different part of the body by constriction and dilation of vessel walls. As the bloods circulate through the body, it might come across vessel blockage. Such blockage may tend to make the heart to exert more pressure in order to keep the blood in circulation. High blood pressure is the main cause to hypertension which can result in heart attack, heart failure, brain stroke etc.

It is therefore of great importance to explore more quantitative and qualitative approach so as to remedy the disease described above. This in turn culminated in a mathematical approach which could predict and provide solution to the problem above.

In the past there has been mathematical models developed for blood flow. H., Yao et al (2000) provides a good computational modeling of blood flow through curved stenosed arteries. W., Kathleen (2003) developed a model on human blood flow measurement and modeling. Jacobson et al (2002) described a model on sausage-string appearance of arteries and arterioles. His study reveals that under certain conditions, the cylindrical shape of a blood vessel may be unstable which can lead to exhibiting constriction and dilation of the vessel.

Procedure

Blood flow from a region of high pressure to region of low pressure except in certain situations. As blood flows to an organ, it is controlled by the constriction and dilation of vessel walls. Reduction in blood flow in the system due to resistance causes the heart to exert more pressure to meet with the amount of blood for circulation by inducing high blood pressure. Within vessels, there exist thin layers of blood which doest not move in contact with the wall of the vessel. The other layers within the vessels have a low and high velocity respectively, with stream centre having greatest velocity. The blood flow in the vessel is laminar. Laminar flow occurs at velocities up to a critical velocity. The flow is turbulent above this velocity.

Turbulence probability is related to the diameter of the vessel and blood viscosity is expressed as

$$R_e = \frac{\rho D v}{\eta}$$

Where R_e is the Reynolds Number

D= Diameter of the tube concerned

η = Viscosity

V= velocity of the fluid

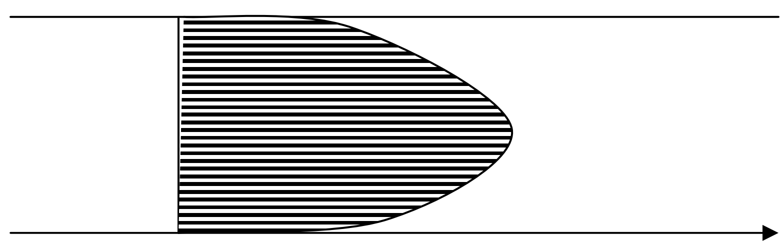

Fig. 1 vessel wall

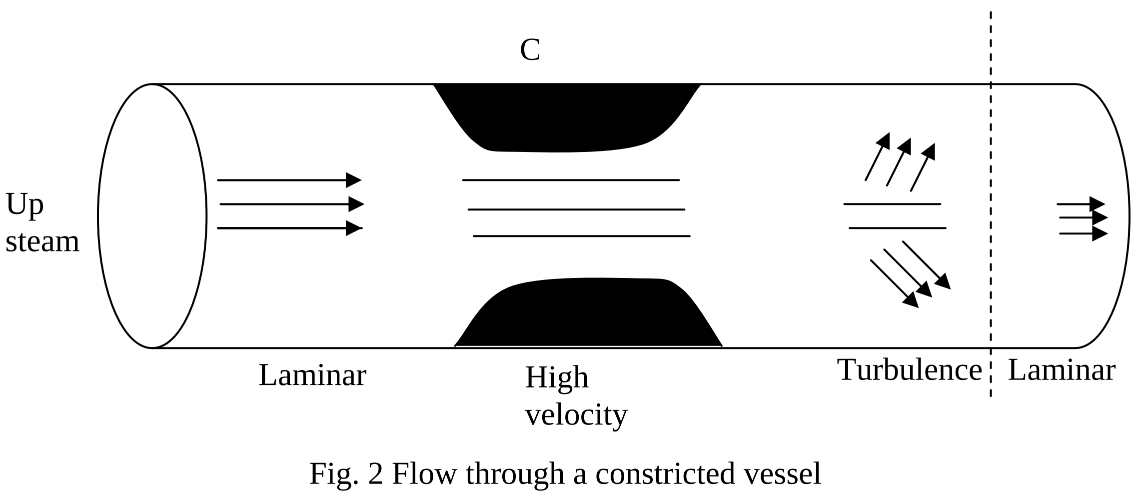

Fig. 2 Flow through a constricted vessel

Basic Assumptions and Formulation of the Model Equation

- The flow is steady and incompressible
- The flow is laminar and turbulent in constricted vessel
- The fluid is Newtonian
- The length of the blood vessel is constant

As we have said earlier, the flow is laminar. Laminar depends on R_e which depends on *v* and *D*. The higher value of R_e, the higher the turbulence probability.

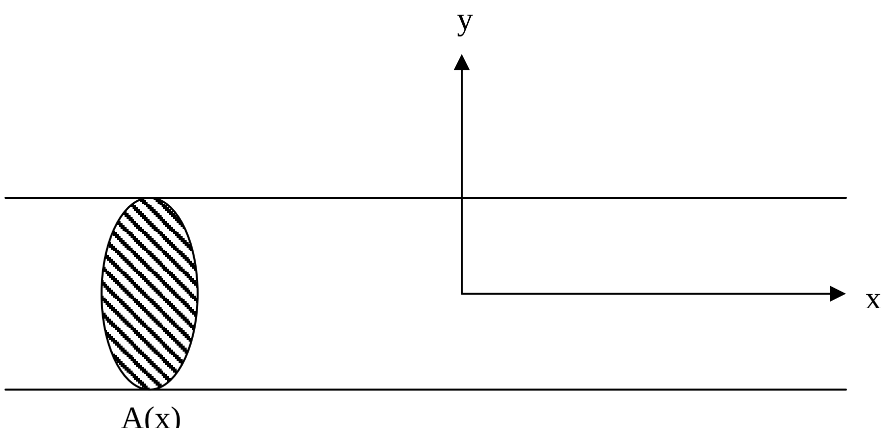

Fig. 3

From fig.3 above, consider a small cross section $A(X)$ of the vessel, and let *J* be the volume flow rate and p_e the excess pressure (given by the difference between normal pressure and hydrostatic pressure).

The motive here is to obtain an in viscid momentum equation describing the flow of blood within the unconstricted distensible vessel. So at a given point X along the length of the vessel and at a given time t, the flow rate *J* , is

$$J = \frac{xA}{t} \quad (1) \quad \Rightarrow \quad \frac{\delta J}{\delta x} = \frac{A}{t} \quad (2) \quad \text{Where } A = \frac{Jt}{x} \quad (3)$$

$$\Rightarrow \quad \frac{\delta A}{\delta t} = \frac{J}{x} \quad (4)$$

Equation (2) and (4) gives $A = t\frac{\delta J}{\delta x}$ (5) and $J = x\frac{\delta A}{\delta t}$ (6)

Hence $$\frac{x}{t} \cdot A = x\frac{\partial A}{\partial t} \quad (7)$$

Substituting (5) into equation (7) we have $\frac{x}{t}\left(t\frac{\partial J}{\partial x}\right) = x\frac{\partial A}{\partial t}$

$$\Rightarrow \frac{\partial A}{\partial t} = -\frac{\partial J}{\partial x} \quad (8)$$

which is the equation of mass conservation in the tube (vessel)
From Area of linear expansivity

$$\frac{\partial A}{A} \propto \partial p_e$$

where ∂A = change in Area, A=Area and

p_e is the presssure $\Rightarrow \frac{\partial A}{A} = \Delta \partial p_e$ (9)

Where Δ is the proportionality constant known as distensibility.
So equation (9) is the equation for the distensibility of the tube.
So from equation (9) $\partial A = A\Delta\partial p_e$ (10)

and

$\partial A = -\frac{\partial J}{\partial x}\partial t$ (11) $\Rightarrow A\Delta\partial p_e = -\frac{\partial J \partial t}{\partial x}$ $\Rightarrow A\Delta\frac{\partial p_e}{\partial t} = -\frac{\partial J}{\partial x}$ (12)

Pressure $p_e = \rho \text{gh}$ (13)

Where $\rho = density$ g=acceleration and h=distance

So $p_e = \rho\frac{uh}{t}$ $\Rightarrow p_e = \rho\frac{ux}{t}$ (14)

from (14) $\frac{\partial p}{\partial x} = \rho\frac{u}{t}$ (15) and from (14) $u = \frac{p_e}{\rho x}t$ (16) $\Rightarrow$

$\frac{\partial u}{\partial t} = \frac{p_e}{\rho x}$ $\Rightarrow \frac{\partial u}{\partial t} = \frac{\rho u x}{\frac{t}{\rho x}}$ $\Rightarrow \frac{\partial u}{\partial t} = \frac{\rho u x}{\rho x t} = \frac{u}{t}$ $\Rightarrow \frac{\partial u}{\partial t} = \frac{u}{t}$ (17)

So $u = t\frac{\partial u}{\partial t}$ (18)

From equation (15) $u = t\frac{\partial p_e}{\rho\partial x}$ (19)

(18) and (19) gives $t\frac{\partial u}{\partial t} = t\frac{\partial p_e}{\rho\partial x}$ $\therefore \rho\frac{\partial u}{\partial t} = -\frac{\partial p_e}{\partial x}$ (20)

Equation (20) is the inviscid momentum equation
When integrated gives

$$\rho\frac{\partial J}{\partial t} = -A\frac{\partial p_e}{\partial x} \quad (21)$$

From equation (12)

$$\partial J = -A\Delta\frac{\partial p_e \partial x}{\partial t}$$

Substituting into equation (21) yields an expression for flow of blood.

$$-\rho A\Delta\partial x\frac{\partial p_e}{\partial t(\partial t)}=-A\frac{\partial p_e}{\partial x}$$

$$\frac{\partial p_e}{\partial t^2}=\frac{1}{\rho\Delta}\frac{\partial p_e}{\partial x^2}$$

$$\Rightarrow\frac{\partial^2 p_e}{\partial t^2}=\frac{1}{\rho\Delta}\frac{\partial^2 p_e}{\partial x^2}$$

$$\Rightarrow\frac{\partial^2 p_e}{\partial t^2}=c^2\frac{\partial^2 p_e}{\partial x^2},\qquad c=(\rho\Delta)^{-\frac{1}{2}} \tag{22}$$

The system is described by equation (20) and (22), which describes the model for the flow of blood within the unconstricted distensible tube (Vessel).

If the vessel is non-uniform (i.e has some constriction in it equation (12) can be expressed as

$$\frac{\partial p_e}{\partial t}=-\frac{c}{Y}\frac{\partial J}{\partial x} \tag{23}$$

Equation (21) becomes

$$\frac{\partial J}{\partial t}=-cY\frac{\partial p_e}{\partial x} \tag{24}$$

$$\frac{\partial^2 p_e}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial p_e}{\partial t}\right)$$

$$\Rightarrow\frac{\partial^2 p_e}{\partial t^2}=\frac{\partial}{\partial t}\left(-\frac{c}{Y}\frac{\partial J}{\partial x}\right)$$

$$=-\frac{c}{Y}\frac{\partial}{\partial t}\left(\frac{\partial J}{\partial x}\right)$$

From equation (24)

$$\frac{\partial^2 p_e}{\partial t^2}=-\frac{c}{Y}\frac{\partial}{\partial t}\left(-cY\frac{\frac{\partial p_e}{\partial x}\partial t}{\partial x}\right)$$

$$\frac{\partial^2 p_e}{\partial t^2}=\frac{c}{Y}\frac{\partial}{\partial x}\left(cY\frac{\partial p_e}{\partial x}\right) \tag{25}$$

Where $Y=\dfrac{A}{\rho c}$ is called the admittance

Boundary Conditions:

The thin layer of blood in contact with the wall of the vessel does not move which means that velocity is zero at this point with distanc e zero at any time which gives

U (0,t) = 0

At the upper bound with distance 2π the blood is in contact with the vessel wall making the velocity zero since there is no movement occurring, this gives

U (2π, t) = 0.

ANALYSIS OF THE MODEL EQUATION

Since equations (22) and (25) describe the model for the flow of blood within the uncontricted distensible tube (vessel) and the vessel with constriction in it respectively, solutions to these two equations can be dealt with knowing the nature of blood flow in the vessel.

Ordinarily, the blood flowing in the vessel (blood vessel) sets up a pressure wave that starts from $x = 0$ (systole) to $x = l$ (diastole). At $x = 0$, the value opens where the pressure wave starts due to the opening of the value and come to rest at $x = l$ when the value closes. The closure of the value prevents the falling back of the blood flowing and the pressure sustained is responsible for the flowing of blood down the tissue. At $t = 0$ the aortic value closes and the left ventricular pressure is about 80mmHg at diastole with first derivative of the blood pressure equals zero. Our consideration is just from the distance $x = 0$ to $x = l$ as shown below

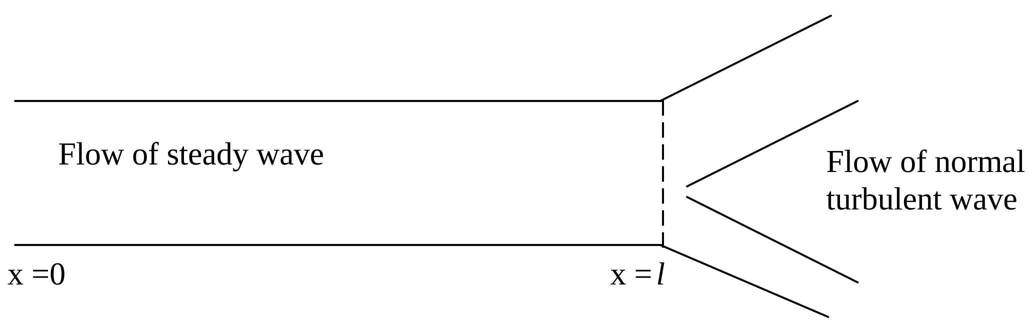

Fig. 4

The position of any blood particle depends on its distance x from one end and on the instant in time. The pressure at any time t can be expressed as
$p_e = f(x,t)$ where x is the distance from $x = 0$ to $x = l$ form the equations (22) and (25) which is wave equation.

In solving the wave equation, the boundary and initial conditions need to be set.

Boundary Conditions

The blood vessel permits flow from $x = 0$ and $x = l$ for all values of time t. So $p_e(x,t)$ becomes

$$\left.\begin{array}{l} p_e(0,t) = 0 \\ p_e(l,t) = 0 \end{array}\right\} \quad \forall\ t \geq 0$$

Initial conditions

Initial pressure at $t = 0$ is therefore $p_e(x,0) = f(x) \leq 80mmHg$ and at that initial condition, the first derivative of the blood pressure is designated by $g(x)$ i.e

$$\left[\frac{\partial p_e}{\partial t}\right]_{t=0} = g(x) = 0$$

Using method of separation of variables and assuming a solution of the form

$$p_e(x,t) = \phi(x)\psi(t) \tag{26}$$

Where

$\phi(x)$ is an arbitrary function of x only

$\psi(t)$ is an arbitrary function of t only

$$\therefore \frac{\partial p_e}{\partial x} = \phi'(x)\psi(t)$$

$$\frac{\partial^2 p_e}{\partial x^2} = \phi''(x)\psi(t)$$

And also

$$\frac{\partial p_e}{\partial t} = \phi(x)\psi'(t)$$

$$\frac{\partial^2 p_e}{\partial t^2} = \phi(x)\psi''(t)$$

Substituting back into equation (22) yields $\phi(x)\psi''(t) = c^2\phi''(x)\psi(x)$

By collecting the like terms we have

$$\frac{\psi''(t)}{\psi(t)} = c^2\frac{\phi''(x)}{\phi'(x)}$$

For the two expressions to be equal for all values of the variables, both expressions must be equal to a constant

$$\frac{\psi''(t)}{\psi(t)} = c^2\frac{\phi''(x)}{\phi'(x)} = constant = k$$

$$\therefore \frac{\psi''(t)}{\psi(t)} = k \quad and \quad \frac{\phi''(x)}{\phi'(x)} = k$$

$$\Rightarrow c^2\phi''(x) - k\phi(x) = 0$$

$$\Rightarrow \phi''(x) - {k}/{c^2}\,\phi(x) = 0$$

And

$\psi''(t) - k\psi(t) = 0$

Considering the boundary conditions on the assumed solution (26) i.e $p_e(x,t) = \phi(x)\psi(t)$

i) $p_e(0,t) = \phi(0)\psi(t)$

ii) $p_e(l,t) = \phi(l)\psi(t)$ $\quad \Bigg\} \quad \forall\ t \geq 0$

Since if $\psi(t) = 0$ then

$p_e(x,t) = 0$ which is a trivial solution, hence we have $\phi(0) = 0$ *and* $\phi(l) = 0$

Now we solve for $\phi''(x) - {k}/{c^2}\,\phi(x) = 0$

Considering different values for k,

for $k = 0$ $\qquad \phi(x) = k_1 + k_2 x$

The solution $p_e(x,t) = \phi(x)\psi(t)$ reduces to zero which gives a trivial solution by applying the boundary conditions.

$\Rightarrow k \neq 0$

For $k > 0$

$$\phi(x) = k_1 \ell^{\frac{\sqrt{k}}{c}s} + k_2 \ell^{\frac{-\sqrt{k}}{c}x}$$

The boundary at $x = 0$ requires

$$k_1 = -k_2$$

The condition at $x = l$ is then impossible to satisfy, hence k 0

$\therefore 0 > k = -\lambda^2$

$\therefore$ our partial differential equation (PDE) becomes

$$\phi''(x) + \frac{\lambda^2}{c^2}\phi(x) = 0 \qquad (27) \text{ and}$$

$$\psi''(t) + \lambda^2\psi(t) = 0 \qquad (28)$$

For (27)

$$\phi''(x) + \frac{\lambda^2}{c^2}\phi(x) = 0$$

The roots becomes $m^2 + {\lambda^2}/{c^2} = 0$

$\Rightarrow m^2 = -\left({\lambda}/{c}\right)^2 \Rightarrow m = \pm {\lambda}/{c}\, i$

$\therefore$ Our solution becomes

$$\phi(x) = A\cos {\lambda}/{c}\, x + B\sin {\lambda}/{c}\, x \qquad (29)$$

For the second PDE (28)

$\psi''(t) + \lambda^2\psi(t) = 0$

For the same value of k also gives the solution

$$\psi(t) = C\cos\lambda t + D\sin\lambda t \qquad (30)$$

$$\therefore\ p_e(x,t) = \left(A\cos {\lambda}/{c}\, x + B\sin {\lambda}/{c}\, x\right)\left(C\cos\lambda t + D\sin\lambda t\right) \qquad (31)$$

Where A, B, C and D are arbitrary constant. The above equation must satisfy the boundary and initial conditions

Applying the boundary condition

$0 = p_e(0,t) = (A\cos(0) + B\sin(0))(C\cos\lambda t + D\sin\lambda t)$

$0 = A(\cos\lambda t + D\sin\lambda t) \ \ \forall\ t \geq 0$

$\Rightarrow A = 0$

$$\therefore\ p_e(x,t) = B\sin {}^{\lambda}\!/_{c}\, x(C\cos\lambda t + D\sin\lambda t) \tag{32}$$

Applying the second boundary condition

$0 = B\sin {}^{\lambda l}\!/_{c}(C\cos\lambda t + D\sin\lambda t)$

$B \neq 0\ \ else\ \ p_e(x,t) = 0$ so

$\Rightarrow \sin {}^{\lambda l}\!/_{c} = 0$

$\therefore\ \dfrac{\lambda l}{c} = n\pi, \quad n \in Z$

$\Rightarrow \lambda = \dfrac{n\pi c}{l}, \quad n \in Z$

Hence the solution to the d.e for $\phi(x)$ *is* $\phi(x) = A\cos {}^{\lambda}\!/_{c}\, x + B\sin {}^{\lambda}\!/_{c}\, x$

The boundary condition at x = 0 gives A = 0

The boundary condition at $x = l$ gives

$\lambda = \dfrac{n\pi c}{l}, \quad n \in Z$

And the pde for the time dependent part is

$\psi(t) = C\cos\lambda t + D\sin\lambda t$

The infinite values of λ gives a corresponding solution for $p_e(x,t)$. Where values of λ are called eigenvalue and each corresponding solution is the eigenfunction. Note that $n \neq 0$ since this will result to $p_e(x,t) = 0$.

Since the wave equation given is linear, if $p_{em}(m = 1,2,\ldots)$ are solution then $p_e(x,t) = \sum_{m=1}^{\infty} p_{em}$ is also a solution.

$\therefore$ the more general solution is

$$p_e(x,t) = \sum_{m=1}^{\infty} p_{em} = \sum_{m=1}^{\infty}\left\{\sin\frac{m\pi x}{l}\left(C_m\cos\frac{mc\pi t}{l} + D_m\sin\frac{cm\pi t}{l}\right)\right\} \tag{33}$$

To find C_m *and* D_m we apply the initial condition $p_e(x.0) = f(x) \leq 25mmHg \ \ for \ \ 0 \leq x \leq l$ substituting in the general solution

$$p_e(x,0) = f(x) = \sum_{m=1}^{\infty}\left\{\sin\frac{m\pi x}{l}(C_m\cos(0) + D_m\sin(0))\right\} = \sum_{m=1}^{\infty} C_m\sin\frac{m\pi x}{l} \tag{34}$$

At t = 0

$$\left[\frac{\partial p_e}{\partial t}\right]_{t=0} = g(x) = 0 \qquad for \ \ 0 \leq x \leq l$$

Differentiating general solution w.r.t t and putting t = 0 gives

$$g(x) = \frac{c\pi}{l}\sum_{m=1}^{\infty} D_m m \frac{\sin m\pi x}{l} \qquad (35)$$

By applying knowledge of Fourier series techniques the coefficient C_m *and* D_m can be known

$$C_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx, \qquad 0 \le x \le l \qquad (36)$$

And also

$$D_m = \frac{2}{m\pi c}\int_0^l g(x)\sin\frac{m\pi x}{l}dx, \qquad 0 \le x \le l \qquad (37)$$

The solution to the problem is thus

$$p_e(x,t) = \sum_{m=1}^{\infty}\left\{(\frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx)\cos\frac{m\pi ct}{l}\sin\frac{m\pi x}{l} + \left(\frac{2}{mc\pi}\int_0^l g(x)\sin\frac{m\pi x}{l}dx\right)\sin\frac{m\pi ct}{l}\sin\frac{m\pi x}{l}\right\}$$

(38a)

Since $f(x) \le 80mmHg$ *and* $g(x) = 0$ thus,

$$p_e(x,t) = \sum_{m=1}^{\infty}\{(\frac{2}{l}\int_0^l 80\sin\frac{m\pi x}{l}dx)\cos\frac{m\pi ct}{l}\sin\frac{m\pi x}{l}\} \qquad (38b)$$

In solving equation (25), the same approach is used, the only difference is depicted by the figure shown below. Since the cause of the constriction is disease and this is foreign to the body system as it is expressed by the equation.

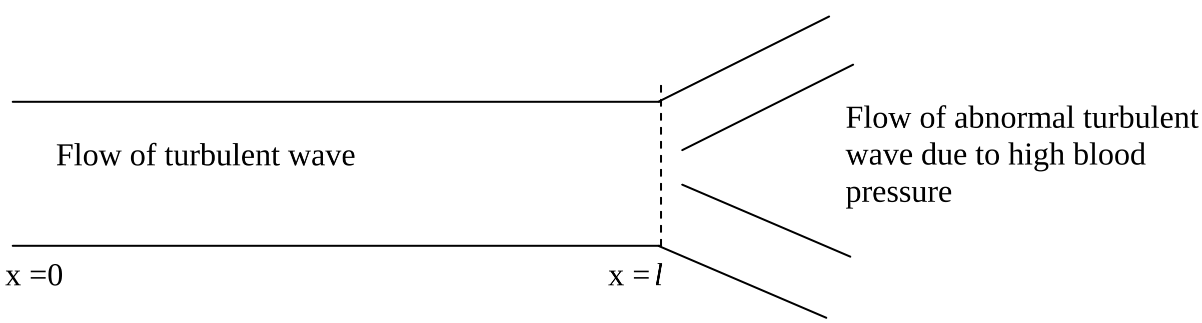

Fig. 5

$$\frac{\partial^2 p_e}{\partial t^2} = \frac{c}{Y}\frac{\partial}{\partial x}\left(cY\frac{\partial p_e}{\partial x}\right)$$

$$\frac{\partial^2 p_e}{\partial t^2} = \frac{c^2}{Y}Y\left(\frac{\partial^2 p_e}{\partial x^2}\right) = c^2\frac{\partial^2 p_e}{\partial x^2}$$

Although, the admittance is put into consideration thus $\frac{c^2}{Y} = a^2$ and $a^2Y = b^2$

Substituting the constant back into the equation yields

$$\frac{\partial^2 p_e}{\partial t^2} = b^2\frac{\partial^2 p_e}{\partial x^2} \qquad (39)$$

Thus the solution is

$$p_e(x,t)=\sum_{m=1}^{\infty}\left\{\left(\frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx\right)\cos\frac{mb\pi t}{l}\sin\frac{m\pi x}{l}+\left(\frac{2}{m\pi b}\int_0^l g(x)\sin\frac{m\pi x}{l}dx\right)\sin\frac{m\pi bt}{l}\sin\frac{m\pi x}{l}\right\} \quad (40a)$$

Since $f(x)\le 80mmHg\ \ and\ \ g(x)=0$ thus,

$$p_e(x,t)=\sum_{n=1}^{\infty}\left\{\left(\frac{2}{l}\int_0^l 80\sin\frac{m\pi x}{l}dx\right)\cos\frac{mb\pi t}{l}\sin\frac{m\pi x}{l}\right\} \quad (40b)$$

Summary and Conclusion

In this study, we have developed mathematical models that describe the velocity and pressure profiles in the human blood vessels. We have considered two situations viz; healthy unconstricted distensible vessel and diseased constricted vessel.

In building these models, we are inspired by the series of assumptions as: (i) The flow is steady and incompressible (ii) The flow is laminar and as well turbulent in constricted vessel (iii) The fluid (blood) is Newtonian (iv) The length of the blood vessel is constant.

From this, we see that the blood flow rate depends on the area of the vessel and the velocity of blood flow, change in area is accompanied by change in blood flow rate. This means that the longer the length of the vessel, the greater would be the resistance and the lesser would be the flow rate since the relationship between flow rate and resistance is given as $J=\frac{Change\ in\ pressure}{\mathrm{Re}\,sis\tan ce}$

Velocity of flow is also inversely proportional to the pressure, that is to say, that increase in blood velocity will bring about the reduction in pressure.

Two solutions for the model equations were obtained using Fourier analysis. In both cases, the excess pressure is a function of constants ‘c’ and ‘b’ respectively. The values of both constants are determined by the value of the distensibility (Δ) of the vessel which is a property of elasticity of blood vessels. In the first case which describes the model for the flow of blood through the uncosntricted distensible vessel, the value of the distensibility of the vessel was high thereby making the constant ‘c’ smaller and in turn the value of the excess pressure also become smaller. While in the case II, which describes the model for the blood flow in constricted vessel, the value of the distensibility of the vessel is low thereby making the value of the constant ‘b’ and the excess pressure high. The low distensibility of the vessel was due to the debris and cholesterol deposited in the blood vessel which reduces the elasticity of the vessel making it unable to expand and accommodate the volume of the blood pumped from the heart and thereby making the heart work harder which leads to high blood pressure (Hypetension).

In conclusion, the excess pressure in constricted blood vessel is higher than the excess pressure in an unconstricted vessel due to the distensiblity of the vessel.

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