African Scientist Vol. 11, No. 1 March 31, 2010
Printed in Nigeria

1595-6881/2010 $12.00 + 0.00

http://www.klobex.org/afs

AFS 2009142/11108

A new method for adding a parameter to a family of bivariate exponential and Weibull distributions

S. M. Umar

Department of Mathematical Sciences, Bayero University, P.M.B. 3011, Kano, Nigeria

(Received December 12, 2009)

ABSTRACT: A new way of introducing a parameter to expand a family of exponential and Weibull distributions is introduced and studied. It is applied to yield a new two-parameter extension of the bivariate exponential distribution which may serve as a competitor to such commonly used two-parameter family of the life distribution as the Weibull, gamma and lognormal distributions. The general method is applied to give a new three-parameter Weibull distribution. The families of distributions expanded by the method introduced have the property that the minimum of a geometric number of independent random variables with common distribution in the family has a distribution again in the family.

Keywords: Bivariate geometric distribution; Geometric extreme stability; Life distribution; Parametric family.

1. Introduction

As stated by Ali, Mikhail and Haq (1978), exponential and Weibull distributions play important roles in analyses of life time or survival data. These are partly because of their convenient statistical theory, their important property of lacking memory as well as partly because of their constant hazard rates. In the situations where the one-parameter family of exponential distribution is not sufficiently broad, a number of wider families such as gamma, Weibull and Gompartz-Makeham distributions are in commonly use, instead. These families of distribution as well as their usefulness are discussed in detail by Cox and Oakas (1984). More complete treatments of each of these families of distributions can also be found in Johnson, Kotz and Balakrishnan (1994).

According Marshall and Olkin (1985), various methods can be used to introduce new parameters in order to expand families of distributions for either adding flexibility or to construct either covariate or correlation models. Introduction of a scale parameter leads to the accelerated life model, and taking powers of the bivariate survival function introduces a parameter that leads to the proportional hazards model. For example, in Weibull (1951) and Feller (1968), it is stated that the family of Weibull distributions contains the exponential distributions and it is constructed by taking powers of exponentially distributed random variables. The family of gamma distributions also contains the exponential distributions and it is constructed by taking powers of the lap lace transform of the exponentially distributed random variables. Arnold (1975) and more recently, Marshall and Olkin (1997), introduced and presented a method of adding parameter to a family of univariate exponential distributions in order to expand it and make it more flexible distribution. In Hawkes (1972) and subsequently Marshall and Olkin (1997), the families of Weibull and gamma distributions were expanded by adding new parameters to it. Some properties of the new families of these distributions are also given in Marshall and Olkin (1997, 1988).

Email Address: surajoumar@yahoo.com

In this paper, a general method of introducing a parameter into a family of bivariate distributions is presented and discussed. In particular, starting with a bivariate survival function $\overline{F}$, the one-parameter family of bivariate survival function

$$\overline{H}(x,y;\alpha)=\frac{\alpha\overline{F}(x,y)}{1-\overline{\alpha}\overline{F}(x,y)}$$

$$=\frac{\alpha\overline{F}(x,y)}{F(x,y)+\alpha\overline{F}(x,y)}, \qquad -\infty<x,y<\infty \qquad 0<\alpha<\infty. \qquad (1.1)$$

where $\overline{\alpha}=1-\alpha,$ is proposed, introduced and discussed in section 2 of this write up. It worth noting that whenever $\alpha=1,\ \overline{H}=\overline{F}.$

The particular case that $\overline{F}$ is an exponential distribution provides a new two-parameter family of distributions that may sometimes be a competitor to the usual bivariate Weibull and gamma families of distributions. This extended family is introduced and presented in section 3. Section 4 gives the method used to derive a three-parameter version of the weibull family of distributions. All the commonly used methods of introducing an additional parameter have a stability property. That is, if the method is applied twice, nothing new is obtained the second time around. Therefore, a power of an exponential random variables have a Weibull distribution, but the power of a Weibull random variables is nothing but another Weibull random variables. Similarly, if in (1.1) above, a bivariate survival function of the form $\overline{H}$ is introduced for $\overline{F}$, then the equation (1.1) gives nothing new. This stability property and the derivation of equation (1.1) is presented and fully discussed in section 5 of this paper. Conclusion of the whole paper is considered in section 6 of this write up.

2. Bivariate density and Hazard rate of the new family

The bivariate survival function $\overline{H}$ given (1.1), always have easily-computed bivariate densities as long as the bivariate functions $\overline{F}$ has a bivariate density function. In particular, if $\overline{F}$ has a bivariate density $f(x,y)$ and hazard rate $r_{\overline{F}}$, then the bivariate survival function $\overline{H}$ has the bivariate density function $h(x,y)$ given by:

$$h(x,y;\alpha)=\frac{\alpha f(x,y)}{\left\{1-\overline{\alpha}\overline{F}(x,y)\right\}^2}$$

$$=\frac{\alpha f(x,y)}{\left\{1-(1-\alpha)\overline{F}(x,y)\right\}^2}$$

$$=\frac{\alpha f(x,y)}{\left\{1-\overline{F}(x,y)+\alpha\overline{F}(x,y)\right\}^2}=\frac{\alpha f(x,y)}{\left\{F(x,y)+\alpha\overline{F}(x,y)\right\}^2} \qquad (2.1)$$

and the corresponding hazard rate is given by:

$$
\begin{aligned}
r(x,y;\alpha) &= \frac{1}{\left\{1-\overline{\alpha}\overline{F}(x,y)\right\}} r_F(x,y) \\
&= \frac{1}{1-(1-\alpha)\overline{F}(x,y)} r_F(x,y) \\
&= \frac{1}{1-\overline{F}(x,y)+\alpha\overline{F}(x,y)} r_F(x,y) = \frac{1}{F(x,y)+\alpha\overline{F}(x,y)} r_F(x,y)
\end{aligned} \tag{2.2}
$$

Therefore,

$$\lim_{x\to-\infty}\lim_{y\to-\infty} r(x,y;\alpha) = \lim_{x\to-\infty}\lim_{y\to-\infty} \frac{r_F(x,y)}{\alpha}.$$

Similarly,

$$\lim_{x\to\infty}\lim_{y\to\infty} r(x,y;\alpha) = \lim_{x\to\infty}\lim_{y\to\infty} r_F(x,y).$$

Based on the result obtained in equation (2.2) and Genest, Ghoudi and Rivest (1995), we can establish the following:

$$\frac{r_F(x,y)}{\alpha} \le r(x,y;\alpha) \le r_F(x,y), \qquad -\infty < x,y < \infty,\ \alpha \ge 1 \tag{2.3}$$

$$r_F(x,y) \le r(x,y;\alpha) \le \frac{r_F(x,y)}{\alpha}, \qquad -\infty < x,y < \infty,\ \alpha \le 1 \tag{2.4}$$

Also,

$$\overline{F}(x,y) \le \overline{H}(x,y;\alpha) \le \overline{F}^{1/\alpha}(x,y), \qquad -\infty < x,y < \infty, \qquad \alpha \ge 1 \tag{2.5}$$

and

$$\overline{F}^{1/\alpha}(x,y) \le \overline{H}(x,y;\alpha) \le \overline{F}(x,y), \qquad -\infty < x,y < \infty, \qquad \alpha \le 1 \tag{2.6}$$

From the same equation (2.2) above, we can see that $\frac{r(x,y;\alpha)}{r_F(x,y)}$ is an increasing function in x and y for $\alpha \ge 1$ and a decreasing function in both x and y for $0 < \alpha \le 1.$

If $F(0,0) = 0,$ the hazard rate $r(0,0;\alpha)$ at the origin of bivariate survival function behaves quite differently then it does for the Weibull or gamma distributions; for both these families, the distribution can be an exponential distributions, or $r(0,0) = 0,\ or\ r(0,0) = \infty,$ so that $r(0,0)$ is discontinuous in the shape parameter. This is not the case with the bivariate family having hazard rates as given in (2.2). Thus, the bivariate family may be useful to make the bivariate function $F(x,y)$ clearer.

However, in spite of equations (2.3) and (2.4) above, it need not be that bivariate function $F(x,y)$ and the bivariate survival function $H(x,y)$ are at all similar to each other.

3. A new family containing the bivariate Exponential Distributions

Consider the bivariate function $\overline{F}(x,y) = exp(-\lambda x - \lambda y),$ the two-parameter family of bivariate survival function

$$\overline{H}(x,y;\alpha,\lambda)=\frac{1}{(\alpha-1)+e^{(\lambda x+\lambda y)}}, \qquad (x>0, y>0, \alpha>0, \lambda>0) \qquad (3.1)$$

is obtained from equation (1.1). The special case $\alpha=\lambda=1$ gives the bivariate exponential distribution. When $\alpha=\lambda\geq 1,$ this bivariate distribution is the conditional bivariate distribution, given $Z>0,$ of a random variable Z with the bivariate logistic survival function $P(Z>z)=\frac{\alpha}{\left(1-(1-\alpha)e^{\lambda x+\lambda y}\right)}$ for $-\infty<z<\infty.$

Considering equation (3.1) above as a special case of (2.1) and (2.2), it can be shown that the bivariate survival $H(x,y)$ has the bivariate density function h given by

$$h(x,y;\alpha,\lambda)=\frac{\alpha\lambda e^{-\lambda x-\lambda y}}{\left[1-(1-\alpha)e^{-\lambda x-\lambda y}\right]^2}=\frac{\alpha\lambda e^{\lambda x+\lambda y}}{\left[e^{\lambda x+\lambda y}-(1-\alpha)\right]^2}, \qquad (x>0, y>0; \alpha>0, \lambda>0),$$

and the corresponding hazard rate

$$r(x,y;\alpha,\lambda)=\frac{\lambda}{1-(1-\alpha)e^{-\lambda x-\lambda y}}=\frac{\lambda e^{\lambda x+\lambda y}}{e^{\lambda x+\lambda y}-(1-\alpha)}, \qquad (x>0, y>0; \alpha>0, \lambda>0).$$

Note that $r(x,y;1,\lambda)=\lambda,$ that is $r(x,y;\alpha,\lambda)$ is decreasing function in both x and y for $0<\alpha\leq 1,$ and that $r(x,y;\alpha,\lambda)$ is an increasing function in x and y for $\alpha\geq 1.$

From (2.3) and (2.4), it follows that

$$\frac{\lambda}{\alpha}\leq r(x,y;\alpha,\lambda)\leq\lambda, \qquad (-\infty<x,y<\infty,\ \alpha\geq 1), \qquad (3.2)$$

$$\lambda\leq r(x,y;\alpha,\lambda)\leq\frac{\lambda}{\alpha}, \qquad (-\infty<x,y<\infty,\ 0\leq\alpha\leq 1), \qquad (3.3)$$

$$e^{-\lambda x-\lambda y}\leq\overline{H}(x,y;\alpha,\lambda)\leq e^{\frac{-\lambda x-\lambda y}{\alpha}}, \qquad (-\infty<x,y<\infty,\ \alpha\geq 1), \qquad (3.4)$$

$$e^{\frac{-\lambda x-\lambda y}{\alpha}}\leq\overline{H}(x,y;\alpha,\lambda)\leq e^{-\lambda x-\lambda y}, \qquad (-\infty<x,y<\infty,\ 0\leq\alpha\leq 1). \qquad (3.5)$$

As it was stated in Barlow and Proschan (1975), that distribution with an increasing hazard rate is new better than used. Similarly, distribution with a decreasing hazard rate is new worse than used. It follows that, when bivariate random variables X and Y have the bivariate distribution H, the conditional bivariate survival function satisfies

$$P(X>x+t, Y>y+t / X>x, Y>y)\begin{cases}\leq P(X>t,Y>t), & (\alpha\geq 1)\\ \geq P(X>t,Y>t), & (0<\alpha\leq 1).\end{cases}$$

Proposition 3.1: The bivariate function $\log h(.,.;\alpha,\lambda)$ is convex for $0<\alpha\leq 1$ and concave for $\alpha\geq 1.$

This result can be verified by differentiating the bivariate function $\log h(.,.;\alpha,\lambda)$ twice with respect to both variables X and Y. Of course, this means that, for $\alpha\leq 1$, the bivariate density function $h(x,y)$ is decreasing and, for $\alpha\geq 1$, $h(.,.;\alpha,\lambda)$ is unimodal, with the mode of each of the two variables given as:

$$\text{mod} = \begin{cases} 0, & (\alpha \le 2), \\ \lambda^{-1}\log(\alpha-1), & (\alpha \ge 2). \end{cases}$$

It follows from equations (3.4) and (3.5) that function H has finite moments of all positive orders. Direct computation shows that, if the two variables have distribution function $\overline{H}(x, y; \alpha, \lambda)$, then each of the two variables has first moment given as

$$E[.] = -\frac{\alpha \log \alpha}{\lambda(1-\alpha)} \qquad (3.6)$$

Note that this quantity is always positive. More generally, for the marginal distribution of random variable X, we have

$$E\left[X^r\right] = r\int_0^\infty \overline{H}(x;\alpha,\lambda)x^{r-1}dx = \frac{r\alpha}{\lambda^r}\int_0^1 \left\{\frac{(-\log q)^{r-1}}{1-(1-\alpha)q}\right\}dq, \qquad (3.7)$$

which of course for r=1, it yields equation (3.6). In the same way, for the marginal distribution of random variable y, the r^{th} moment is as given in equation (3.7) above with Y replacing X.

The lap lace transform of marginal distribution h of each of two random variables X and Y can also be obtained as follows. For the random variable X, it is follows as:

$$E\left[e^{-s\lambda x}\right] = \int_0^1 \left\{\frac{\alpha q^s}{(1-(1-\alpha)q)^2}\right\}dq. \qquad (3.8)$$

Similarly, that of random variable Y can be obtained in the same way by replacing X with Y.

Both (3.7) and (3.8) can be expressed as infinite series whenever $|1-\alpha| \le 1.$ From this, the integrands of (3.7) and (3.8) can be expanded in a power series and the result be integrated term by term to give:

$$E\left[X^r\right] = \tfrac{r\alpha}{\lambda^r}\int_0^\infty x^{r-1}e^{-x}\sum_{j=0}^\infty \alpha^{-j}e^{-jx}dx = \tfrac{r\alpha}{\lambda^r}\sum_{j=0}^\infty \tfrac{\alpha^{-j}\Gamma(r)}{(j+1)^r} \qquad (|1-\alpha| \le 1),$$

and

$$E\left[e^{-sX}\right] = \alpha\int_0^1 q^s \sum_{j=0}^\infty (j+1)q^j\alpha^{-j}dq = \alpha\sum_{j=0}^\infty \alpha^{-j}\,\tfrac{j+1}{s+j+1} \qquad (|1-\alpha| \le 1). \qquad (3.9).$$

That of marginal distribution of random variable Y can also be obtained in the same by using the corresponding moment and lap lace transform of the random variable Y.

According to Karlin, Proschan and Barlow (1961) and as a consequence of proposition 3.1, total positivities properties yield moment inequalities that are not generally true. In particular, the coefficient of variation $\frac{\delta}{\mu}$ is less than 1 when $\alpha > 1$ and is greater 1 for $\alpha < 1.$ δ^2 is the variance while μ is the first moment of random variables X or Y. It is easy to show that the k^{th} quartile x_k of $\overline{H}$ is given by: $x_k = \frac{1}{\lambda}\log\left(\frac{(1-k)+\alpha k}{1-k}\right).$

The median of each of the random variables X and Y is also given by: Median of X=Median of Y= $\dfrac{\log(1+\alpha)}{\lambda}.$

It can be seen that median, mode and expectations of random variable X or Y is an increasing function in α and decreasing function in the scale parameter λ.

From the monotonic of $\log x,$ $\log y$ and the fact that $\log x \le x-1$ and $\log y \le y-1$ as well as the values of random variables X and Y are both positive, it follows that

$\text{mod}\,e(X) \le med(X) \le \frac{\alpha}{\lambda} \le E(X)$ and also $\text{mod}\,e(Y) \le med(Y) \le \frac{\alpha}{\lambda} \le E(Y)$, but note that

$\lim_{\alpha\to\infty}\frac{\mathrm{mod}\,e(X)}{E(X)}=\lim_{\alpha\to\infty}\frac{\mathrm{mod}\,e(Y)}{E(Y)}=1.$ When $E(X)$ and $E(Y)$ are fixed say equal 1, then the weak limit of $\overline{H}$ as α tends to infinity is degenerate at 1, and the limit as α tends to zero is degenerate at point zero. Notice also that $\lim_{x\to\infty}\lim_{y\to\infty} r(x,y;\alpha,\lambda)=\lambda$ is bounded and continuous in the parameters, like gamma distribution but unlike the Weibull distribution.

4. Extension bivariate Weibull Distributions

Given that

$$\overline{F}(x,y)=e^{-(\lambda x)^{\beta}-(\lambda y)^{\beta}}, \quad x\ge 0, y\ge 0; \beta>0 \qquad (4.1)$$

is a bivariate Weibull survival function, then equation (1.1) gives the new three-parameter survival function

$$\overline{H}(x,y;\alpha,\lambda,\beta)=\frac{\alpha e^{-(\lambda x)^{\beta}-(\lambda y)^{\beta}}}{1-(1-\alpha)e^{-(\lambda x)^{\beta}-(\lambda y)^{\beta}}}. \qquad (4.2)$$

This geometric –extreme stable extension of the bivariate Weibull distribution may sometimes be a competitor to the more usual three-parameter Weibull distribution with survival function

$$\overline{F}(x,y;\alpha,\lambda,\delta)=Exp\left\{-\lambda(x-\delta)-\lambda(y-\delta)\right\}^{\beta}, \quad x\ge\delta, y\ge\delta; \lambda>0, \beta>0; -\infty<\delta<\infty.$$

If X and Y have a bivariate exponential distribution with parameter 1, then $\frac{X^{1/\beta}}{\lambda}$ and $\frac{Y^{1/\beta}}{\lambda}$ have the survival function (4.1). Similarly, if X and Y have the survival function (3.1) with parameter $\lambda=1$, then $\frac{X^{1/\beta}}{\lambda}$ and $\frac{Y^{1/\beta}}{\lambda}$ have the survival function (4.2). Hence, moments of (4.2) can be obtained from non integer moments of (3.1). Therefore, from (3.6), it follows that, if X and Y have the bivariate survival function (4.2), then

$$E\left[X^{s}Y^{s}\right]=\tfrac{x^{\alpha}}{\beta}.\tfrac{y^{\alpha}}{\beta}\sum\tfrac{(1-\alpha)^{j}}{(j+1)^{\frac{s}{\beta}}}\Gamma\left(\tfrac{s}{\beta}\right), \qquad |1-\alpha|\le 1. \qquad (4.3)$$

If $|1-\alpha|>1,$ then the moments can be obtained from equation (3.4) by applying change of variable technique that was earlier used in getting equation (4.3). However, those moments can not be given in closed form; thus, even the first moment of (4.2) must be obtained numerically. By writing

$E\left[X^{s}Y^{s}\right]=\int_0^\infty\int_0^\infty sx^{s-1}y^{s-1}\overline{F}(x,y)\,dxdy, \qquad s>0,$ it can be shown that $\lim_{\beta\to\infty}E\left[X^{s}Y^{s}\right]=\lambda^{-s}, \qquad s>0.$ Of course, these are moments of random variables degenerate at $\frac{1}{\lambda}$.

It should be noted that the density and hazard rate of the distribution given by the equation (4.2), can be directly obtained from (2.1) and (2.2). In particular, the hazard rate is given by

$$r(x,y;\alpha,\lambda,\beta)=\frac{\lambda\beta(\lambda x)^{\beta-1}(\lambda y)^{\beta-1}}{\left[1-(1-\alpha)e^{-(\lambda x)^{\beta}-(\lambda y)^{\beta}}\right]}.$$

This function can be verified using elementary calculus that this hazard rate is increasing if $\alpha\ge 1, \beta\ge 1$ and decreasing if $\alpha\le 1, \beta\le 1$. If $\beta>1,$ then the hazard rate is initially increasing and eventually increasing, but there may be one interval where it is decreasing. Similarly, if $\beta<1,$ then the hazard rate is initially decreasing and eventually decreasing, but there may be one interval where it is increasing. At those intervals, the slope changes are subtle and not easy to be seen graphically.

5. Geometric-Extreme stability of bivariate Distribution

If $X_1, X_2,...$ and $Y_1, Y_2,...,$ are sequences of independent identically distributed bivariate random variables with distributions in the family (1.1), and if N has a geometric distribution on {1,2,3,…}, then minimum and maximum of both $(X_1, X_2,..., X_n)$ and $(Y_1, Y_2,..., Y_n)$ also have distributions in the family. To understand why this property may be of interest, recall that extreme value distributions are limiting distributions for extrema, and as such they are sometimes usefull approximation. In practice, a random variable of interest may be the extreme of only a finite, possibly random, number N of random variables. When N has a geometric distribution, the random variable has a particularly nice stability property, just like that of extreme value distributions.

Suppose that N is independent of $X_1, X_2,...$ and $Y_1, Y_2,...,$ with a geometric (p) distribution, that is $P(N=n)=(1-p)^{n-1}p, \qquad n=1,2,3,...,$ and let

$$U_1=\min(X_1, X_2,..., X_n), \qquad V_1=\max(X_1, X_2,..., X_n)$$
$$U_2=\min(Y_1, Y_2,..., Y_n), \qquad V_2=\max(Y_1, Y_2,..., Y_n) \tag{5.1}$$

Definition: *If* $F\in \mathrm{T}$ *implies that the distributions of* $U_i(V_i)$, $(i=1,2)$ *are in* T*, then* T *is said to be geometric-minimum stable (geometric-maximum stable). If* T *is both geometric-minimum and geometric-maximum stable, then* T *is said to be geometric-extreme stable.*

The term \`maximum-geometric stable\` has already been used by Rachev and Resnick (1991) and subsequently by Marshall and Olkin (1997) to describe a related but more restricted concept. They apply the term not to families of distributions but to individual distributions; in their sense, a distribution is \` maximum-geometric stable\` if the location-scale parameter family generated by the distribution is geometric-maximum stable in our sense. The two ideas essentially coincide for families T that are parameterized by location and scale. Most of the families considered in this paper are not of that form, a notable exception being the logistic distribution. For instance the family of logistic distributions, with bivariate survival functions of the form $\overline{F}(x,y)=\dfrac{1}{1+\theta e^{\lambda x+\lambda y}}, \quad -\infty<x,y<\infty;\ \theta,\lambda>0,$ is a geometric-extreme stable family; indeed, distributions in this family are geometric-extreme stable even in the sense of Rachev and Resnick (1991). The fact that this family is geometric-minimum stable was utilized by Arnold (1989) to construct a stationary process with logistic marginal.

For random variables U_1 and U_2 of equation (5.1),

$$\overline{H}(x,y)=P(U_1>x,U_2>y)=\sum_{n=1}^{\infty}\overline{F}^n(x,y)(1-p)^{n-1}p$$
$$=\frac{p\overline{F}(x,y)}{1-(1-p)\overline{F}(x,y)}, \qquad -\infty<x,y<\infty. \tag{5.2}$$

As an extension of univariate parametric family of distributions given by Marshall and Olkin (1997), the bivariate parametric family of distributions given by the equation (5.2), is geometric-minimum stable.

For the random variables V_1 and V_2 , also defined in (5.1), arguments similar to those used above show that $H(x,y)=P(V_1\le x,V_2\le y)=\dfrac{pF(x,y)}{1-(1-p)F(x,y)}, \qquad -\infty<x,y<\infty,$

so that

$$\overline{H}(x,y)=\frac{p\overline{F}(x,y)}{1-(1-p)\overline{F}(x,y)}, \qquad -\infty<x,y<\infty. \tag{5.3}$$

Based on Marshall and Olkin (1997), it can also be shown here that the bivariate parametric family stated in equation (5.3) is geometric-maximum stable.

The bivariate families defined in (5.2) and (5.3) above, combine nicely to form a single parametric family $\xi = \xi\left(F(X,Y)\right) = \left\{H\left(x,y;\alpha\right), \alpha > 0\right\}$, where $\overline{H}\left(x,y\right)$ is given by equation (1.1); in equation (5.2), $0 < \alpha = p \le 1$, and, in (5.3), $\alpha = \frac{1}{p} \ge 1$. It also worth noting that $\overline{H}\left(x,y;1\right) = \overline{F}\left(x,y\right)$, therefore, $F\left(x,y\right) \in \xi;$ further more, it is also true that $H\left(x,y;\alpha\right)$ is stochastically increasing in α.

Proposition 5.1: *The parametric family ξ of distributions of the form (1.1) is geometric-maximum stable.*

Proof. To verify this proposition, it is enough to verify closure of ξ under a kind of composition, as follows. Suppose that $\overline{H}\left(x,y\right) = \dfrac{\kappa \overline{H}\left(x,y;\alpha\right)}{\left\{1-\left(1-\kappa\right)\overline{H}\left(x,y;\alpha\right)\right\}}$, where $H\left(x,y;\alpha\right)$ is given by (5.3).

Then $\overline{M}\left(x,y\right) = \dfrac{\kappa\alpha \overline{F}\left(x,y\right)}{\left\{1-\left(1-\alpha\kappa\right)\overline{F}\left(x,y\right)\right\}}$. This shows that $M\left(x,y\right) \in \xi$, and, consequently, ξ has both geometric-maximum and geometric-minimum stability.□

The proof of proposition (5.1) also shows that, if F is replaced by any other distribution in ξ , then that distribution will also generate ξ.

Some facts concerning geometric-extreme stable families are evident and may be worth noting: the same properties also hold for geometric-minimum and geometric-maximum stable families.

(a) If T_1 and T_2 are geometric-extreme stable families, then $\mathrm{T}_1 \cup \mathrm{T}_2$ and $\mathrm{T}_1 \cap \mathrm{T}_2$ are geometric-extreme stable families; the empty set is vacuously such a family.

(b) Every distribution F determines a geometric-extreme stable family $\mathrm{T}\left(F\right)$. If $H \in \mathrm{T}\left(F\right)$, then $\mathrm{T}\left(H\right) = \mathrm{T}\left(F\right)$. Thus, the minimal geometric-extreme stable families form a partition of the set of all distributions into a set of equivalence classes. Here, a minimal geometric-extreme stable family is a family which is nonempty and has no nonempty geometric-extreme stable subfamily.

(c) If F and H differ only by a scale (location) parameter, then $\mathrm{T}\left(H\right)$ can be obtained from $\mathrm{T}\left(F\right)$ by a common scale (location) change.

(d) Suppose that $F \in \mathrm{T}$ implies that $\overline{F}\left(0\right) > 0$, and define F_+ by

$$\overline{F}_+\left(x,y\right) = \begin{cases} 1, & x \le 0, y \le 0, \\ \frac{\overline{F}(x,y)}{\overline{F}(0)}, & x \ge 0, y \ge 0. \end{cases}$$

If F is geometric-extreme stable, then $\left\{F_+ : F \in \mathrm{T}\right\}$ is geometric-extreme stable.

(e) Let F be a family of distribution functions, and let

$$\mathrm{T}_{\theta,\delta} = \left\{H : H\left(x,y\right) = F^{\theta}\left(x-\delta, y-\delta\right) \quad for \quad some \quad F \in \mathrm{T}\right\}.$$

If T is geometric-extreme stable, then $\mathrm{T}_{\theta,\delta}$ is geometric-extreme stable for all $\theta > 0$ and all real δ.

5.1 Use of Geometric distribution in extreme stability property

The geometric-extreme stability property of $\xi = \xi\left(F\right)$ is rather remarkable, and it depends upon the fact that a geometric sum of independent identically distributed geometric random variables has a geometric distribution. This partially explains why random-minimum stability cannot be expected if the geometric distribution is replaced by some other distribution on $\left\{1,2,...\right\}$. Thus, if the above development is repeated, for instance with the assumption that N-1 has a Poisson distribution, then ξ would be replaced by a family that would not be Poisson-extreme stable.

If F is a distribution function and $\overline{H}(x,y;\theta)=\sum_{n=1}^{\infty}\overline{F}^{n}(x,y)\,p_n(\theta)$ has the stability property then the discrete distribution must satisfy the functional equation

$$\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}z^m p_m(\theta)\right\}^n p_n(\alpha)=\sum_{n=1}^{\infty}z^n p_n(\kappa), \qquad 0\le z\le 1.$$

Under certain regularity conditions, the only solution to this equation is the geometric distribution.

6. Conclusion

It can be concluded here that the general method of introducing one-parameter into a family of bivariate distribution is developed and presented. The extended exponential distribution provide a new method of adding two-parameter to a family of bivariate distribution which may sometimes compete with bivariate Weibull and gamma families of distributions. Another method of derivation of three-parameter version of Weibull family of distribution is presented. It is also stated in this paper that all the methods of adding parameter to a different families of different distributions commonly possessed stability properties.

ACKNOWLEDGEMENTS: The Author thanks all those that contributed positively with suggestions that led to the improved version of this paper.

References

Ali, H.M, Mikhail, N.N and Haq, M.S (1978): A class of bivariate distributions including the bivariate logistic. J. Mult. Ana. 8: 405-512.

Arnold, B.C (1975): Multivariate exponential distributions based on hierarchical successive damage. J. Appl. Prob. 12: 142-147.

Arnold, B.C (1989): A logistic Process constructed using geometric minimization. Statist. Prob. Lett. 7: 253-259.

Barlow, R.E and Proschan, F (1981): Statistical theory and Reliability and Life Testing. New York: Hol, USA.

Cox, D.R and Oakes, D (1984): Analysis of survival data. London: Chapman and Hall.

Downton, F (1970): Bivariate exponential distributions in reliability. J.R. Statist. Soc. B 32: 403-417.

Feller, W (1968): An introduction to probability Theory and its Applications, 3rd ed. New york, Wiley, USA.

Genest, C, Ghoudi, K and Rivest, L.P (1995): A semi parametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika 82: 543-552.

Hawkes, A.G (1972): A bivariate exponential distribution with applications to reliability. J.R. Statist. Soc. B 34: 129-131.

Johnson, N.I, Kotz, S and Balakrishnan, N (1994): Continuous Univariate Distributions, 1, 2nd ed. New york, Wiley, USA.

Karlin, S, Proschan, F and Barlow, R.E (1961): Moment inequalities of Polya frequency functions. Pacific J. Math. 11: 1023-1033.

Marshall, A.W and Olkin, I (1985): A family of bivariate distributions generated by the bivariate Bernoulli distribution. J. Amer. Statist. Assoc. 80: 332-338.

Marshall, A.W and Olkin, I (1988): Families of multivariate distributions. J. Amer. Statist. Assoc. 83: 834-841.

Marshall, A.W and Olkin, I (1997): A new method for adding a parameter to a Family of distributions with application to the exponential and Weibull families. Biometrika. 84: 641-652

Rachev, S.T and Resnick, S (1991): Maximum-geometric infinite divisibility and stability. Commun. Statist. Stoch. Mod. 7: 191-218.

Weibull, W (1951): A Statistical distribution of wide applicability. J. Appl. Mech. 18: 293-297.