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On the hydrodynamic interactions between two spheres in stokes flow

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ABSTRACT: This paper examines the interactions between two spheres in an unbounded fluid. Using bispherical coordinates, no-slip and far-field boundary conditions, an exact solution of Stokes equations for the translational motion of two spheres of arbitrary size and arbitrary orientation with respect to their directions of motion are obtained. This solution is in form of truncated infinite series. The various hydrodynamic forces exerted on the spheres are calculated. The results from the force calculations show that when the two spheres are in close proximity, greater number of terms has to be retained in the series before convergence is achieved.

Keywords: Bispherical coordinates, Stokes flow, Hydrodynamic.

1 Introduction

A body moving in a viscous fluid generates long range flow disturbances which influences the motion of other bodies in the flow. This mutual influence via the fluid is called hydrodynamic interaction. This is of particular interest in various disciplines such as physics, colloid chemistry and weather forecasting and other application areas (see [3, 4, 12]). There has been a great interest in recent years concerning inter particle forces especially in the field of electric and magnetic fields (see [14, 16]). The understanding of many-particle systems begins with the systematic study of simpler systems. The two-sphere system can be used as bases for studying inter particles forces. The description of the hydrodynamic interaction of two spheres in terms of linear Navier-Stokes equations for steady, incompressible flow (creeping-flow equations or Stokes flow equations) brings about substantial mathematical simplification of the flow field. In many situations the creeping-flow equations provide an adequate approximation to the actual motion. This approach was used by many authors to study the hydrodynamic interactions between two spheres in Stokes flow. Such studies includes; [13] who considered the flow in the vicinity of two spheres rotating about their line of centres. The motion of two spheres in a shear field was investigated by [11]. Davies (1969) analyzed the translation and rotation of two unequal spheres in a viscous fluid. Others are; [5] considered the flow past ensemble of particles via numerical simulations. The theory of forces between two spheres was studied theoretically using dipole by [7, 8]. The hydrodynamic drag force on two spheres was also studied via numerical simulations by [18]. The solution for the equation of motion of a viscous sphere in the presence of interracial slip was derived by [12]. All the solutions, however, are highly involved and require extensive numerical calculations to arrive at explicit results. So that now the translation and rotation of two spheres about an axis with no-slip boundary conditions on the spheres is known. The corresponding problem for spheres with arbitrary sizes and orientation with respect to their directions of motion has not received much attention. For this reason, a general recurrence relation satisfying the Stokes equation for arbitrary translation direction is unavailable in literature. This general form of recurrence relation has been derived in this paper. The methodology employed is using bispherical coordinates, no-slip and far-field boundary conditions, and an exact solution of Stokes equations for the translational motion of two spherical bodies of arbitrary size and orientation with respect to their directions of motion are obtained. The various hydrodynamic forces exerted on each sphere are calculated.

2 Description of the problem

Suppose we wish to determine the velocity field in the vicinity of two spheres moving with arbitrary velocities in an incompressible Newtonian fluid. We shall denote the local fluid velocity and pressure fields by $\mathbf{v}$ and P, respectively. We assumed that flow is sufficiently slow for $\mathbf{v}$ to satisfy the Stokes equations

$$\mu\nabla^2\mathbf{v} = \nabla P, \tag{2.1}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{2.2}$$

where μ is the dynamic viscosity of the fluid. We further assumed that the no-slip applies on the surfaces the spheres and that $\mathbf{v} \to 0$ far away from the spheres. The creeping flow equations (equations (2.1) and (2.2) represent a substantial simplification of the over all flow field and have a number of advantages: Analytical solutions of the equations are possible for some relatively complex geometry. Also because of their linearity, the superposition principle can be used to add solutions together.

2.1 Geometry of the problem.

Geometrically, the problem is depicted as in Figure 1. The geometric quantities needed to describe the problem and for stating the boundary conditions are also shown on the Figure. The spheres are denoted by *I* and *II*, with radii denoted by R_I and R_{II} respectively. The distance between their centres is denoted by *d*. The inherent bispherical nature of the problem suggests that it can best be tackled using bispherical coordinate system (ξ,η,ϕ) in which the governing equations and the boundary conditions can be accommodated relatively easily. This coordinate system has its origin on the spheres line of centres at the point which divides d into unequal parts. In terms of the cylindrical coordinates (ρ, z, ϕ) , the bispherical coordinates are given by [3]

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \qquad z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \qquad \phi = \phi. \tag{2.3}$$

By restricting the range of the coordinates to

$$0 \le \eta \le \pi, \qquad -\infty < \xi < \infty, \qquad 0 \le \phi \le 2\pi, \tag{2.4}$$

the coordinate system is unique (see [6]). In this work, we shall assumed that the sphere above the z = 0 (see Figure 1) corresponds to $\xi = \xi_I$ with radius $\mathrm{R_I}$ and the sphere in the lower half plane corresponds to $\xi = -\xi_{II}$ with radius $\mathrm{R_{II}}$ and that both ξ_I, ξ_{II} are positive. The centre to centre separation between the two spherical bodies will be denoted by

$$\mathrm{d} = \mathrm{h_I} + \mathrm{h_{II}} \tag{2.5}$$

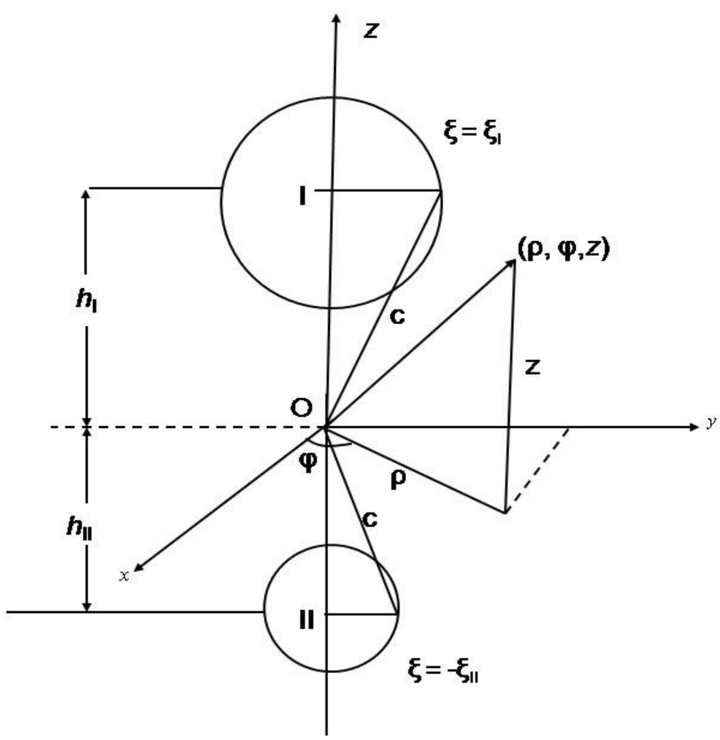

Figure 1 Schematic diagram of the geometry of two sphere problem.

where $h_{I,II}^2 = R_{I,II}^2 + c^2$ (see Figure 1). Thus, the centre to centre distance between the two spheres can be written as

$$d = \sqrt{R_I^2 + c^2} + \sqrt{R_{II}^2 + c^2}. \tag{2.6}$$

After eliminating the square roots from (1.6) we obtained,

$$c = \frac{1}{2d}\left(\left(d^2 - (R_I^2 + R_{II}^2)\right)^2 - 4R_I^2 R_{II}^2\right)^{\frac{1}{2}}, \quad d \geq R_I + R_{II}. \tag{2.7}$$

3 Analysis for spheres of arbitrary size and speeds

The pressure field of the Stokes equation satisfies Laplace's equation (see [1]) and the velocity field, being a particular solution of (2.1), also satisfies Laplace's equation. A natural link provided by equation (2.3) between bispherical and cylindrical coordinates allows one to derive the solution to Laplace's equation in terms of the former from the latter. In cylindrical coordinates (ρ, z, ϕ) equation (2.1) can be written as (see [1])

$$\frac{\partial}{\partial \rho}\left(\frac{P}{\mu}\right) = (\nabla^2 - \frac{1}{\rho^2})v_\rho - \frac{2}{\rho^2}\frac{\partial v_\phi}{\partial \phi}, \tag{3.1}$$

$$\frac{\partial}{\rho \partial \phi}\left(\frac{P}{\mu}\right) = (\nabla^2 - \frac{1}{\rho^2})v_\phi + \frac{2}{\rho^2}\frac{\partial v_\rho}{\partial \phi}, \tag{3.2}$$

$$\frac{\partial}{\partial z}\left(\frac{P}{\mu}\right) = \nabla^2 v_z, \tag{3.3}$$

where the Laplacian operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \tag{3.4}$$

and v_ρ, v_ϕ and v_z are components of the velocity field in cylindrical coordinates. Hence the continuity equation (2.2) can be written as,

$$\frac{\partial v_\rho}{\partial \rho} + \frac{1}{\rho}v_\rho + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0. \tag{3.5}$$

In cylindrical coordinates, the pressure and the components of the velocity field of the Stokes equation satisfy the following general form (see [11])

$$P = \frac{\mu}{c}\sum_{k=0}^{\infty}(\bar{W}_k^0 \sin(k\phi) + \bar{W}_{-k}^0 \cos(k\phi)) \tag{3.6}$$

$$v_\rho = \frac{1}{2}\sum\nolimits^{\infty}{}_{k=0}[(\frac{\rho}{c}\bar{W}_k^0 + W^1{}_k + W^{-1}{}_k)\sin(k\phi) + (\frac{\rho}{c}\bar{W}_{-k}^0 + W^1{}_{-k} + W^{-1}{}_{-k})\cos(k\phi)] \tag{3.7}$$

$$v_z = \frac{1}{2}\sum\nolimits^{\infty}{}_{k=0}[(\frac{z}{c}\bar{W}_k^0 + 2W_k^0)\sin(k\phi) + (\frac{z}{c}\bar{W}_{-k}^0 + 2W_{-k}^0)\cos(k\phi)] \tag{3.8}$$

$$v_\phi = \frac{1}{2}\sum\nolimits^{\infty}{}_{k=0}[(W_k^{-1} - W_k^1)\cos(k\phi) - (W_{-k}^{-1} - W_{-k}^1)\sin(k\phi)] \tag{3.9}$$

where W_m^i expressed explicitly in terms of bispherical coordinates are given by

$$W_m^i = \Delta^{\frac{1}{2}}\sum_{n=|m|+i}^{\infty}[A_{mn}^i \cosh((\frac{2n+1}{2})\xi) + B_{mn}^i \sinh((\frac{2n+1}{2})\xi)]P_n^{|m|+i}(\cos(\eta)) \tag{3.10}$$

and

$$\bar{W}_m^0 = \Delta^{\frac{1}{2}}\sum_{n=|m|+i}^{\infty}[\bar{A}_{mn}^i \cosh((\frac{2n+1}{2})\xi) + \bar{B}_{mn}^i \sinh((\frac{2n+1}{2})\xi)]P_n^{|m|+i}(\cos(\eta)). \tag{3.11}$$

$P_n^m(\mu)$ are associated Legendre polynomials of order n and rank m define by

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_n(\mu) \tag{3.12}$$

$$\Delta = \cosh(\xi) - \cos(\eta). \tag{3.13}$$

Here, i m and n are integers. The next task is to determine the various constants $A_{mn}^i, \bar{A}_{mn}^i, B_{mn}^i, \bar{B}_{mn}^i$ so that equations (2.1) and (2.2) are satisfied.

3.1 Determination of coefficients from the continuity equation.

Differentiating equations (3.7) through (3.9) and substituting the appropriate terms into equation (3.5) and simplifying we obtained the following,

$$(\frac{3}{c} + \frac{\rho}{c}\frac{\partial}{\partial\rho} + \frac{z}{c}\frac{\partial}{\partial z})\bar{W}_k^0 + (\frac{k+1}{\rho} + \frac{\partial}{\partial\rho})W_k^1 + (\frac{\partial}{\partial\rho} + \frac{1-k}{\rho})W_k^{-1} + 2\frac{\partial W_k^0}{\partial z} = 0, \tag{3.14}$$

and

$$(\frac{3}{c}+\frac{\rho}{c}\frac{\partial}{\partial\rho}+\frac{z}{c}\frac{\partial}{\partial z})\bar{W}^{0}_{-k}+(\frac{k+1}{\rho}+\frac{\partial}{\partial\rho})W^{1}_{-k}+(\frac{\partial}{\partial\rho}+\frac{1-k}{\rho})W^{-1}_{-k}+2\frac{\partial W^{0}_{-k}}{\partial z}=0. \tag{3.15}$$

In order to simplify the problem, we derived the following relations for transforming equation (3.14) into bispherical coordinates

$$\frac{\partial}{\partial z}=\frac{1}{c}(-\cosh(\xi)\cos(\eta)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\xi}-\sinh(\xi)\sin(\eta)\frac{\partial}{\partial\eta}), \tag{3.16}$$

$$\frac{\partial}{\partial\rho}=\frac{1}{c}(\cosh(\xi)\cos(\eta)\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\eta}-\sinh(\xi)\sin(\eta)\frac{\partial}{\partial\xi}). \tag{3.17}$$

Making use of these results in equations (3.14), (3.15) we obtained the continuity equation in bispherical coordinates as,

$$\begin{aligned}&3\bar{W}^{0}_{\pm k}+[(k+1)W^{1}_{\pm k}+(1-k)W^{-1}_{\pm k}]\Delta\csc(\eta)-\cos(\eta)\sinh(\xi)\frac{\partial\bar{W}^{0}_{\pm k}}{\partial\xi}-\sin(\eta)\cosh(\xi)\frac{\partial\bar{W}^{0}_{\pm k}}{\partial\eta}\\&+(\cosh(\xi)\cos(\eta)-1)(\frac{\partial W^{1}_{\pm k}}{\partial\eta}+\frac{\partial W^{-1}_{\pm k}}{\partial\eta}-2\frac{\partial W^{0}_{\pm k}}{\partial\xi})-\sin(\eta)\sinh(\xi)(\frac{\partial W^{-1}_{\pm k}}{\partial\xi}+\frac{\partial W^{1}_{\pm k}}{\partial\xi}+2\frac{\partial W^{0}_{\pm k}}{\partial\eta})=0.\end{aligned} \tag{3.18}$$

Numerical values of constants satisfying the continuity equation in the form given by equations (3.14) and (3.15) depends upon the value of k Different values of k will give rise to different continuity equation. Determination of the coefficients requires that equations (3.14) and (3.15) are simplified to obtain a series of recurrence relations. It is cumbersome for one to go over the simplifications whenever a different value of k is desired. One important contribution made here is the derivation of general recurrence relation valid for $k \geq -1$.

3.2 Recurrence relations from the continuity equation

The constants $\bar{A}^{0}_{\pm kn}, \bar{B}^{0}_{\pm kn}, A^{-1}_{\pm kn}, B^{-1}_{\pm kn}, A^{1}_{\pm kn}, B^{1}_{\pm kn}, A^{0}_{\pm kn}$ and $B^{0}_{\pm kn}$ in the auxiliary solutions given in equations (3.14) and (3.15) are determine explicitly by using the continuity equation and the boundary conditions. To begin with the recurrence relations for general k (see equation (3.18)) shall be derived and later use some particular values to calculate the forces and to characterize the resulting flow field. To make the task easier, exponential rather than hyperbolic functions shall be used. Now define,

$$\Lambda_{n}(A,B,\xi)=A^{i}_{\pm kn}e^{(n+\frac{1}{2})\xi}+B^{i}_{\pm kn}e^{-(n+\frac{1}{2})\xi} \quad i=0,-1,1. \tag{3.19}$$

Making use of the auxiliary function solutions given by equations (3.10) and (3.11) in equation (3.18) and after lengthy simplifications we obtained,

$$
\begin{aligned}
0 &= 5\bar{A}^{0}_{\pm kn} + 2(k-n)\bar{A}^{0}_{\pm k(n-1)} + (n+1+k)\bar{A}^{0}_{\pm k(n+1)} + 2A^{-1}_{\pm kn} - A^{-1}_{\pm k(n+1)} - A^{-1}_{\pm k(n-1)} \\
&+2(n+1+k)(-n+k)A^{1}_{\pm kn} + (n+2+k)(n+1+k)A^{1}_{\pm k(n+1)} + 2(1+2n)A^{0}_{\pm kn} \\
&+2(k-n)A^{0}_{\pm k(n-1)} - 2(n+1+k)A^{0}_{\pm k(n+1)} + (k-n+1)(-n+k)A^{1}_{\pm k(n-1)},
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
0 &= 5\bar{B}^{0}_{\pm kn} + 2(k-n)\bar{B}^{0}_{\pm k(n-1)} + (n+1+k)\bar{B}^{0}_{\pm k(n+1)} + 2B^{-1}_{\pm kn} - B^{-1}_{\pm k(n+1)} - B^{-1}_{\pm k(n-1)} \\
&+2(n+1+k)(k-n)B^{1}_{\pm kn} + (k-n+1)(k-n)B^{1}_{\pm k(n-1)} + (n+2+k)(n+1+k)B^{1}_{\pm k(n+1)} \\
&-2(2n+1)B^{0}_{\pm kn} + 2(n-k)B^{0}_{\pm k(n-1)} + 2(n+1+k)B^{0}_{\pm k(n+1)}.
\end{aligned}
\tag{3.21}
$$

Equations (3.20) and (3.21) are the general recurrence relations involving the constants that most be satisfied to ensure (2.2) holds. However, by themselves they are insufficient to determine the actual values of the constant A's and B's. In order to achieve that we must examine the additional constraints that arise from the no slip boundary conditions acting on the surfaces of the spheres.

4 Recurrence Relations Arising From the Boundary Conditions

For the problem consider here (translational motion of two spheres in an unbounded quiescent fluid), it is necessary to fragment the problem into a number of cases viz: the k = 1 and k = 0 modes. The k =1 mode corresponds to an arbitrary translational motion along the x and y plane whilst the k = 0 mode is for translation motion along the line of centre (z axis). Thus, in this formulation, we are considering the truncated form of equations (3.6) to (3.9) with k from 0 to 1. We desire to find the constants (A's and B's) such that equations (3.6) - (3.9) satisfy both the Laplace equation (which means they must satisfy (3.20) and (**Error! Reference source not found.**) above) and the no slip conditions on the surface of the spheres. In addition the velocity field should tend to zero as we moves far away from the spheres. We Consider the k = 1 first.

4.1 Recurrence Relations for the k=1 mode

In cylindrical coordinates, the velocity field is given by (see [15])

$$\mathbf{v} = i_{\rho} v_{\rho} + i_{\phi} v_{\phi} + i_{z} v_{z}. \tag{4.1}$$

Here i_{ρ}, i_{ϕ} and i_{z} are units vectors in the ρ, ϕ and z directions respectively. A particular solution satisfying the boundary condition can be written in terms of the x, y and z components of the velocity fields as,

$$v_{\rho} = U_{x}^{N} \cos(\phi) + U_{y}^{N} \sin(\phi), \tag{4.2}$$

$$v_{\phi} = -U_{x}^{N} \sin(\phi) + U_{y}^{N} \cos(\phi), \tag{4.3}$$

$$\text{v\_z} = 0, \tag{4.4}$$

where, $U^N = \left(U_x^N, U_y^N, U_z^N\right)$ are the translational velocities of sphere $\mathrm{N} = \mathrm{I}, \mathrm{II}$ in the *x, y* and *z*-directions respectively. The linearity of these equations implies that the recurrence relations involving the x and *y* velocity components can be treated separately. In view of this, the formal treatment of translational motion along the *y* direction is presented below. The translational motion along the *x* direction can be treated in an entirely analogous fashion. For a sphere translating with only *y* component of the velocity, the no slip condition on its surface implies that

$$U_y^N = \frac{1}{2}\left(\frac{\rho}{c}\overline{W}_1^0 + W_1^1 + W_1^{-1}\right), \tag{4.5}$$

$$U_y^N = \frac{1}{2}\left(W_1^{-1} - W_1^1\right), \tag{4.6}$$

$$0 = \frac{1}{2}\left(\frac{z}{c}\overline{W}_1^0 + 2W_1^0\right). \tag{4.7}$$

Note that equations (4.5) to (4.7) give 6 sets of equations when evaluated on both spheres. Determination of a recurrence relation for the *k = 1* mode from the boundary conditions can be done in a number of ways.

4.2 Numerical Calculations of the Constants

The problem now is the determination of the sets of coefficients $\{A_{1n}^0\}, \{A_{1n}^{-1}\}, \{A_{1n}^1\}, \{\overline{A}_{1n}^0\}$, $\{B_{1n}^0\}, \{B_{1n}^{-1}\}, \{B_{1n}^1\}, \{\overline{B}^0_{1n}\}$ such that the boundary conditions on the spheres and at infinity together with the continuity equations are satisfied. Since the functions W_1^0, W_1^{-1}, W_1^1 and $\overline{W}_1^0$ are non-singular for $-1 \le \mu \le 1$, then the boundary conditions far away from the spheres are satisfied. For $\xi = \xi_I$ and $\xi = -\xi_{II}$, equations (4.5) - (4.7) are evaluated on both spheres to give six equations. These equations were combined with two equations ((3.20) and (3.21)) from continuity, to get a total of eight linear equations that were solved numerically. Calculations of the coefficients were carried out using FORTRAN and MATLAB programming languages. To achieve that, the infinite series solution given by (3.10) and (3.11) were first truncated and retain only the leading *N* terms. For each *N* algebraic system of equations in the unknowns A_{1n}^0, A_{1n}^1, A_{1n}^{-1}, $\overline{A}_{1n}^0$, B_{1n}^0, B_{1n}^1, B_{1n}^{-1} and $\overline{B}_{1n}^0$, *n* = 0, 1, 2…*N* were solved. In order to achieve good accuracy when the two spheres are in close proximity, a greater number of terms have to be retained in the series. The largest value of *N* used is 125 equivalents to a 1000 by 1000 \$times\$ array. The calculations of the coefficients enable the calculations of the velocity field.

It is ostensibly difficult also to say anything about the convergence of the constants by inspecting the equations. Perhaps the only method of establishing convergence is by numerical means. As suggested by [3] a very useful check for the convergence is that

$$sum_1 = \sum [2n(n+1)A^0{}_{1n} + (2n+1)A_{1n}{}^{-1}], \tag{4.8}$$

$$sum_2 = \sum [2n(n+1)B^0{}_{1n} - (2n+1)B_{1n}{}^{-1}], \tag{4.9}$$

should vanish for *N* sufficiently large.

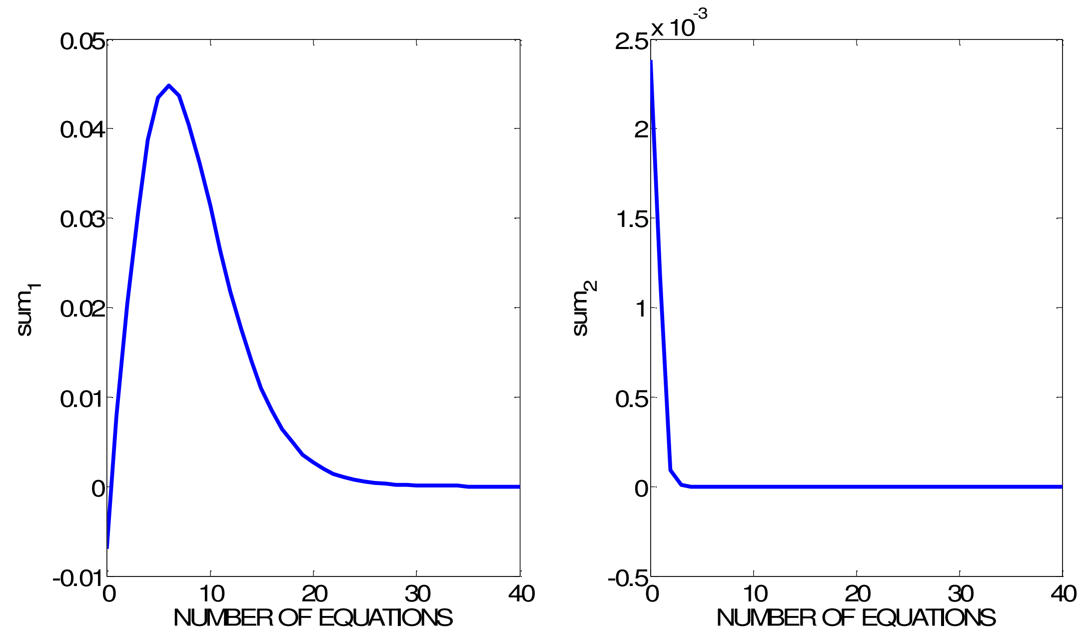

Figure 2 A plot of the convergence test using equations \re{sum1} and \re{sum2} against the number of equations for sphere sizes R_I and $R_{II} = 0.5\times10^{-3}, 10^{-4}$ m respectively, with the following translational velocities $U_y^I = 3\times10^{-3}$, $U_y^{II} = 0$ m s^{-1} . (a) Represent the calculation of the sum of the coefficients using equation 4.8. (b) Represent the calculation of the sum of the coefficients using equation 4.9.

Figure 2 shows typical calculations of sum_1 and sum_2 for sphere sizes R_I and $R_{II} = 0.5\times10^{-3}, \times10^{-4}$ m respectively against the number of equations. It can be discerned from the Figure that the coefficients not only converges, but they do so quickly (small values of *N*). Next we consider the $k = 0$ mode.

4.3 Recurrence Relations for the k=0 mode

The k = 0 mode corresponds to an axisymmetrical flow (see[9]) arising from the translational motion of the two spheres along their line of centres with constant velocity in an otherwise quiescent and unbounded fluid. The no slip boundary condition requires that

$$v_\rho = 0, \tag{4.10}$$

$$v_\phi = 0, \tag{4.11}$$

$$v_z = U_z^N. \tag{4.12}$$

In view of equations (3.7)-(3.9), these conditions imply that

$$0 = \frac{1}{2}(\frac{\rho}{c}\overline{W}_0^0 + W_0^1 + W_0^{-1}), \tag{4.13}$$

$$0 = \frac{1}{2}(W_0^{-1} - W_0^1), \tag{4.14}$$

$$U_z^N = \frac{1}{2}(\frac{z}{c}\overline{W}_0^0 + 2W_0^0). \tag{4.15}$$

Procedure analogous to that used for $k = 1$ mode was followed to obtain the constants arising from $k = 0$ mode.

5 Forces on translating spheres

Consider a body with constant translating velocity in an unbounded fluid. Then the hydrodynamic force on the body is as given by (see [2])

$$\mathbf{F} = \int_{\text{body}} \prod \cdot d\vec{S}, \tag{5.1}$$

where $\prod$ is the pressure stress tensor given by

$$\prod = -p\mathbf{I} + 2\mu\Delta. \tag{5.2}$$

Here I is a tensor and Δ is the rate of deformation tensor. Using equation (5.1**Error! Reference source not found.**), one can derive formulas for calculating the forces on the two sphere in the *x, y* and *z* directions. However, following the analysis of [11] one can write the equations for the force field in this formulation as

$$F_x^{I,II} = -2^{\frac{3}{2}}\pi\mu c \sum\nolimits^{\infty}_{n=0} (A_{-1n}^{-1}) \pm B_{-1n}^{-1}), \tag{5.3}$$

$$F_y^{I,II} = -2^{\frac{3}{2}}\pi\mu c \sum\nolimits^{\infty}_{n=0} (A_{1n}^{-1} \pm B_{1n}^{-1}), \tag{5.4}$$

$$F_z^{I,II} = -2^{\frac{5}{2}}\pi\mu c \sum\nolimits^{\infty}_{n=0} (A_{0n}^{0} \pm B_{0n}^{0}). \tag{5.5}$$

It is to be noted that (+) is to be used for the sphere *I* while (-) is to be used for sphere *II.* All the force calculations in what follows are non-dimensionalised using the Stokes formula (see [1])

$$F = 6\pi\mu R_{I,II} U^{I,II}, \tag{5.6}$$

where $U^{I,II}$ is speed of the spheres. The separations distances (the minimum distance between the spheres surface) are non -dimensionalised by the sum of the spheres radii as

$$d_{r_I,r_{II}} = \frac{d-(R_I+R_{II})}{R_I+R_{II}}. \tag{5.7}$$

6 Results and Discussion

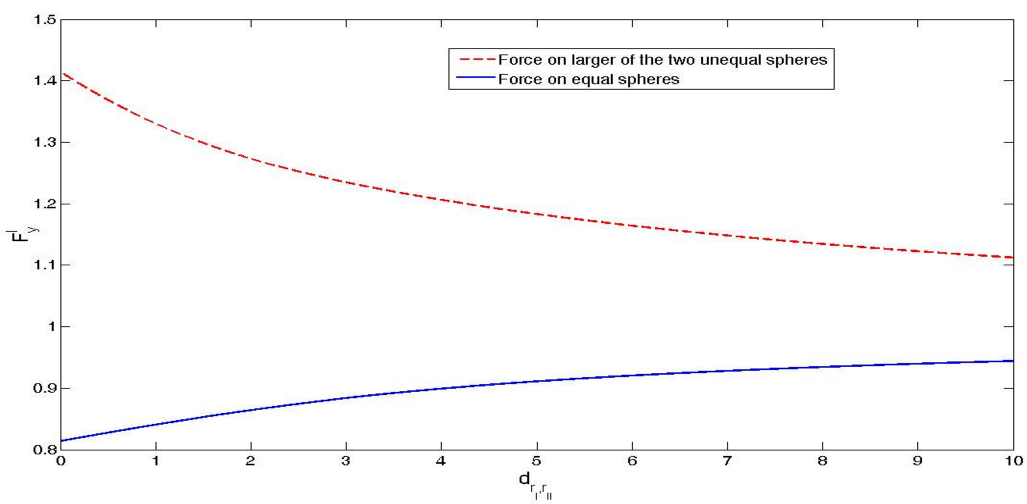

Figure 3 A plot of the calculated *y*-component of the forces for some sphere sizes against the separation distances. The dotted curve shows the forces on the sphere with size $R_I = 5\times10^{-3}$ m and speed $U_y^I = 10^{-3}$ m s^{-1}. This sphere is interacting with another sphere which is ten times smaller and non-swimming. The blue curve shows the corresponding force calculations on one of the spheres when two spheres of equal sizes, $R_I = R_{II} = 5\times10^{-3}$ m and speeds $U_y^I = U_y^{II} = 1\times10^{-3}$ m s^{-1} are interacting. The Figure is plotted using equation (5.4).

A plot of the calculated *y*-component of the forces for some sphere sizes against the separation distances. The dotted (--) line show the forces on the sphere with size $R_I = 5\times10^{-3}$ m and speed $U_y^I = 1\times10^{-3}$ m s^{-1} . This sphere is interacting with another sphere which is ten times smaller and non-swimming. The solid line shows the corresponding force calculations on one of the spheres when two spheres of equal sizes, $R_I = R_{II} = 5\times10^{-3}$ m and speeds $U_y^I = U_y^{II} = 1\times10^{-3}$ m s^{-1} are interacting. The Figure is plotted using equation 5.4 .

Figure3 shows the computed *y*-component of the forces for spheres of equal and unequal sizes and speeds. The equal spheres have sizes $R_I = R_{II} = 5\times10^{-3}$ m and speeds $U_y^I = U^{II}{}_y = 1\times10^{-3}$ m s^{-1}. The sizes of the unequal spheres are $R_I = 5\times10^{-3}$, $R_{II} = 5\times10^{-4}$ m and speeds $U_y^I = 1\times10^{-3}, U_y^{II} = 0$ m s^{-1}. The calculations were done up to 1 separation distance. Subsequently, extrapolation was carried out to estimate the results at close distances. A particular problem associated with the force calculations is that the results are quite erratic as the spheres get very close. Accuracy can be improved by addition of more terms in the system of linear equations (that is by increasing N).

From Figure 3, two contrasting behaviour are demonstrated. For spheres of equal sizes, the force exerted on each sphere as they come closer, gets smaller. It is predicted that, the limiting value of the forces on either sphere is 82% of the corresponding result from Stokes formula (single sphere). However, when the spheres are unequal in size, the force on the larger sphere get bigger as the separations decreases. The prediction in this case, is that the limiting force is about 141% of the corresponding force on a solitary sphere calculated using the Stokes formula. From the point of view of prey perception the interest here is mainly in the case when the spheres are moving head on towards each other. Because in that scenario the various forces on the spheres (or cell bodies) will be greatest. In that case the line along which the spheres are moving towards each other will be taken as the z-axis and limit out investigation to the $k = 0$ mode.

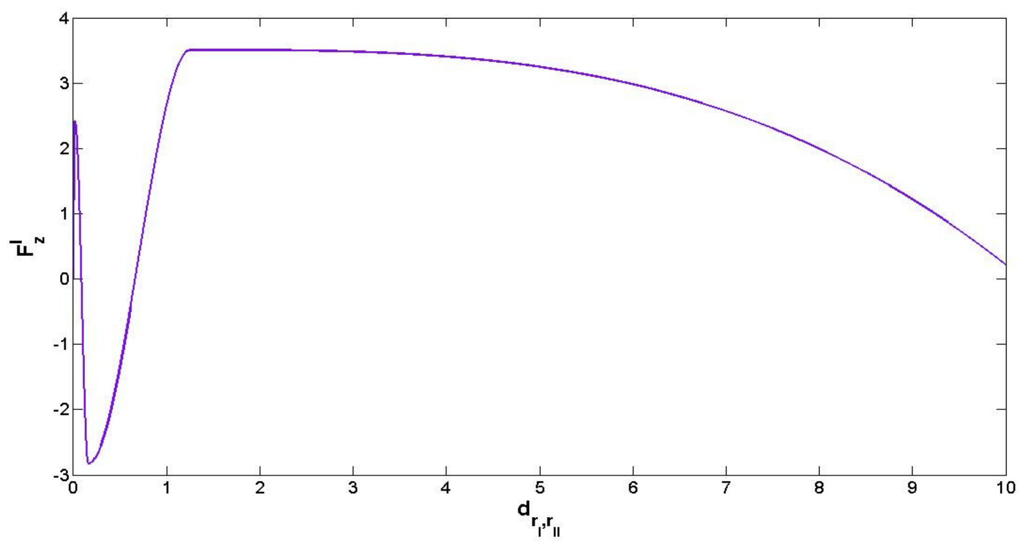

Figure 4 Computed z-component of the forces on unequal spheres

Figure 4 shows a plot z-component of the forces on one of the spheres for the same sphere sizes and speeds shown on Figure 3. Comparing the results from the Figures we can say that the component of the forces on the spheres line of centres (vertical axis) tend to be larger. Generally speaking, the Figure shows that the forces exerted increases as the separation distances decreases. This tends to suggest that the flow field surrounding any one of the spheres will experience greater hydrodynamic disturbances as the spheres get closer. Given that some non-visual planktonic microorganisms can find food, mate or escape from impending predators through changes in the surrounding flow field (see [10,17]), the two sphere problem discussed in this paper can be used to study the process of perception of inert particles by hydromechanical means.

7 Conclusions

In this paper, we looked at the interaction between two spheres of arbitrary sizes and speeds falling in Stokes flow. Using bispherical coordinates, the system of partial differential equations were transformed into a truncated infinite system of linear equations that were solved numerically. These solutions enable the calculations of the velocity field and the hydrodynamical forces on the spheres. The main conclusions are as follows:

1. The accuracy of the computed forces depends upon the separation distances.
2. When the spheres are in close proximity, greater number of terms have to be retained in the series before convergence is achieved.
3. The flow field surrounding one sphere is distorted by the presence of the other. The extent of the distortion depends upon the separation as well as the sphere sizes and speeds.
4. The two sphere system can be used to study interactions between microorganisms such as plankton.

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